21. I. P. Andreichikov and V. I. Yudovich, "Self-oscillatory modes branching from Poiseuille flow in a plane channel," Dokl. Akad. Nauk SSSR, 202, No. 4, 791 (1972).
22. S. K. Godunov, "Numerical solution of boundary-value problems for systems of linear ordinary differential equations," Usp. Mat. Nauk, 16, No. 3(99), 171-174 (1961).
23. S. D. Conte, "The numerical solution of linear boundary-value problems," SIA Review, 8, No. 3, 309-321 (1966).

OSCILLATIONS OF AN IDEAL LIQUID ACTED UPON BY SURFACE-TENSION FORCES. CASE OF A DOUBLY CONNECTED FREE SURFACE

V. R. Orel

UDC 532.593

Many articles have appeared on the problems of small oscillations of an ideal liquid acted upon by surface-tension forces. Oscillations of a liquid with a single free surface are treated in [1, 2]. Oscillations of an arbitrary number of immiscible liquids bounded by equilibrium surfaces on which only zero volume oscillations are assumed possible are investigated in [3]. We consider below the problem of the oscillations of an ideal liquid with two free surfaces on each of which nonzero volume disturbances are kinematically possible. The disturbances satisfy the condition of constant total volume. A method of solution is presented. The problem of axisymmetric oscillations of a liquid sphere in contact with the periphery of a circular opening is considered neglecting gravity. The first two eigenfrequencies and oscillatory modes are found.

§ 1. Suppose a certain volume Q of an ideal liquid bounded by solid walls of a container S and two free surfaces Σ_1 and Σ_2 (Fig. 1) is in a state of stable equilibrium; ρ is the density of the liquid, and σ_1 and σ_2 are the surface tensions. The external field of body forces has the potential Π.

We consider small oscillations of the liquid about the equilibrium position. We denote by $\mathbf{n}_i(\xi)$ the normal to the undisturbed surface Σ_i ($i=1, 2$) at the point ξ directed outward from the region Q, and by $u_i(\xi, t)$ a small displacement along $\mathbf{n}_i$ at time $t \geq 0$. We assume that the displacement $u_i(\xi, t)$ is a twice continuously differentiable function of the parameter $\xi (\in \Sigma_i)$. We denote by D_i the set of such functions. Let $D = D_1 \times D_2$ be the space of all pairs of functions $\{u_1, u_2\}$ where $u_i \in D_i$. We use the vector notation $\mathbf{u} = \{u_1, u_2\}$ for the elements of the set D. We define the scalar product in D ($\mathbf{u}, \mathbf{v} \in D$)

$$(\mathbf{u}, \mathbf{v}) = \int_{\Sigma_1} u_1 v_1 d\Sigma_1 + \int_{\Sigma_2} u_2 v_2 d\Sigma_2.$$

We introduce the displacement potential $\Phi(q, t)$, $q \in Q$ to describe small oscillations of an ideal liquid [4]. For any $t \geq 0$ the potential Φ is a solution of the problem

$$\Delta\Phi = 0, \quad q \in Q; \qquad (1.1)$$
$$\partial\Phi/\partial\mathbf{n}|_S = 0; \quad \partial\Phi/\partial\mathbf{n}_i|_{\Sigma_i} = u_i \quad (i = 1, 2).$$

The necessary condition for the solvability of the inner Neumann problem (1.1) is the conservation of volume [5]

$$(1, \mathbf{u}) = \int_{\Sigma_1} u_1 d\Sigma_1 + \int_{\Sigma_2} u_2 d\Sigma_2 = 0. \qquad (1.2)$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 64-71, May-June, 1976. Original article submitted May 22, 1975.

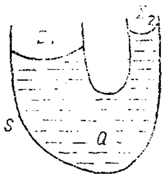

Fig. 1

We denote by D_0 the subspace of the space D which contains the elements **u** satisfying Eq. (1.2). It is known [5] that for every $u \in D_0$ problem (1.1) has a solution Φ which is unique to within an arbitrary function of the time f(t). We set $\Psi_i(\xi, t) \equiv \Phi(\xi, t)|_{\xi\in\Sigma_i}$ (i = 1, 2). It can be shown [1-3] that a matrix of linear operators $G = \|G_{ik}\|$ (i, k = 1,2) exists connecting an arbitrary element **u** of D_0 with the corresponding vector $\Psi = \{\Psi_1, \Psi_2\}$:

$$\mathbf{\Psi} = G\mathbf{u} + f\mathbf{1}. \tag{1.3}$$

Using the properties of harmonic functions [5] it can be established that the operator G is symmetric and positive in the space D_0; this means that ($\mathbf{u} \neq 0$)

$$(G\mathbf{u}, \mathbf{u}) > 0, \ \mathbf{u} \in D_0. \tag{1.4}$$

We consider the matrix differential operator $A = \|A_i\delta_{ik}\|$ (i, k = 1, 2) and D,

$$A_i u_i \equiv \sigma_i(-\Delta_i + \tau_i(\xi))u_i(\xi, t), \quad \xi \in \Sigma_i, \ u_i \in D_i. \tag{1.5}$$

Here Δ_i is the Laplacian operator [6] on the surface Σ_i, and the function $\tau_i(\xi)$ is given by the expression [7]

$$\tau_i(\xi) = \frac{1}{\sigma_i}\frac{\partial\Pi}{\partial\mathbf{n}_i} - 4H_i^2(\xi) + 2K_i(\xi),$$

where $H_i(\xi)$ is the mean curvature and $K_i(\xi)$ is the Gaussian curvature [6] at the given point $\xi \in \Sigma_i$.

Let L_i be the line of intersection of the surfaces Σ_i and S. The kinematic slipping condition for the displacement $u_i(\xi, t)$ on L_i has the form [1, 7]

$$\partial u_i/\partial \mathbf{e}_i + \chi_i(\xi)u_i = 0, \ \xi \in L_i \ (i = 1, 2). \tag{1.6}$$

Here $\mathbf{e}_i$ is a unit vector along the outward normal to the line L_i drawn tangent to the surface Σ_i at the point ξ; the function $\chi_i(\xi)$ is given by the expression [1, 7]

$$\chi_i(\xi) = [\varkappa_i(\xi)\cos\gamma_i - \eta_i(\xi)]/\sin\gamma_i \ (\sin\gamma_i \neq 0),$$

where γ_i is the contact angle, $\varkappa_i(\xi)$ is the curvature of a normal cross section of the free surface Σ_i along the direction $\mathbf{e}_i$; $\eta_i(\xi)$ is the similarly defined curvature of a normal cross section of the wetted surface S at point $\xi \in L_i$ (i = 1, 2).

Let D_i^χ be the set of functions u_i from D_i which satisfies the conditions (1.6). It can be shown [8] that the operators A_i (1.5) are symmetric on the sets D_i^χ (i = 1, 2). Therefore, the matrix operator A is symmetric on $D_1^\chi \times D_2^\chi$. We set $W_0 \equiv D_0 \cup (D_1^\chi \times D_2^\chi)$. Since the volume Q is in stable equilibrium, taking account of the form of the second variation of the potential energy of the system of two surfaces Σ_1 and Σ_2 [9] we obtain ($\mathbf{u} \neq 0$)

$$(A\mathbf{u}, \mathbf{u}) > 0, \ \mathbf{u} \in W_0. \tag{1.7}$$

We write down the linearized dynamic conditions satisfied by the values Ψ_1 and Ψ_2 of the displacement potential Φ on the free surfaces [1, 2, 4] (k = 1, 2)

$$-\rho(\partial^2\Psi_k/\partial t^2)(\xi, t) = A_k u_k(\xi, t) + f(t), \ \xi \in \Sigma_k. \tag{1.8}$$

Assuming everywhere a time dependence of the form $\exp(i\omega t)$ and using (1.3), we write (1.8) in the form (f = const)

$$\omega^2\rho G\mathbf{u} = A\mathbf{u} + f\mathbf{1},$$

from which we obtain by using (1.2), (1.4), and (1.7)

$$\omega^2 = [(A\mathbf{u}, \mathbf{u})/\rho(G\mathbf{u}, \mathbf{u})](> 0), \ \mathbf{u} \in W_0. \tag{1.9}$$

Following [8] we consider the chain of minimization problems

$$\omega_j^2 = \min_{W_{j-1}} \frac{(A\mathbf{u}, \mathbf{u})}{\rho(G\mathbf{u}, \mathbf{u})}, \ j = 1, 2, \ldots, \tag{1.10}$$

to find the eigenfrequencies ω_j and the oscillatory modes $\mathbf{z}_j$ of the volume Q of an ideal liquid, where W_{j-1} (j = 2, 3, ...) is a subspace of space W_0 orthogonal to the vectors $\{\mathbf{z}_1, \ldots, \mathbf{z}_{j-1}\}$, known from the solution of the preceding problems, on which the successive minima $\{\omega_1^2, \ldots, \omega_{j-1}^2\}$ of Eq. (1.10) are reached.

§ 2. We indicate one possible way of constructing and solving the sequence of problems (1.10). We denote by φ_k, ψ_j the eigenfunctions of the operators A_1, A_2 on the sets D_1^χ, D_2^χ, and by ν_k, μ_j (k, j = 1, 2,...) the corresponding eigenvalues

$$A_1\varphi_k = \nu_k\varphi_k, \ \varphi_k \in D_1^\chi; \ A_2\psi_j = \mu_j\psi_j, \ \psi_j \in D_2^\chi. \tag{2.1}$$

Since each operator A_i is symmetric on D_i^χ [8] the eigenvalues ν_k, μ_k (k = 1, 2,...) are real, and the systems of functions $\{\varphi_k\}$, $\{\psi_k\}$ are orthogonal on the corresponding D_i^χ. Without loss of generality we assume that the systems $\{\varphi_k\}$, $\{\psi_k\}$ are normalized and the sets of eigenvalues $\{\nu_k\}$, $\{\mu_k\}$ are arranged in ascending order.

Let N be a positive integer. We consider the finite-dimensional subspace W_0^N of the space $W_0(D_0 \cup (D_1^\chi \times D_2^\chi))$ such that every element $\mathbf{u} \in W_0^N$ has the form

$$\{u_1, u_2\} = \left\{\sum_{k=1}^{N} a_k\varphi_k; \ \sum_{k=1}^{N} b_k\psi_k\right\}.$$

By definition, the elements of the set W_0^N satisfy the conservation of volume (1.2)

$$(\mathbf{u}, 1) = \sum_{k=1}^{N} a_k \int_{\Sigma_1} \varphi_k d\Sigma_1 + \sum_{k=1}^{N} b_k \int_{\Sigma_2} \psi_k d\Sigma_2 = \sum_{k=1}^{2N} \alpha_k w_k = 0, \tag{2.2}$$

where

$$\alpha_{2k-1} = a_k; \ \alpha_{2k} = b_k \ (k = 1, 2, \ldots, N);$$
$$w_{2k-1} = \int_{\Sigma_1} \varphi_k d\Sigma_1, \ w_{2k} = \int_{\Sigma_2} \psi_k d\Sigma_2. \tag{2.3}$$

Knowing the set of numbers (2.3) $\{w_k\}$ we can construct a fundamental set of solutions $Y_N = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of Eq. (2.2). The vectors $\mathbf{y}_k$ entering Y_N are 2N dimensional, and their number n is determined by the number of nonzero coefficients w_k (2.3) and the kinematic restrictions imposed on the oscillations of the volume Q. It can be shown that $2(N-1) \le n \le 2N$. In particular, if displacements u_i with nonzero volumes are admissible on both surfaces, and among the numbers w_k at least two are different from zero, $n = 2N-1$.

To each vector $\mathbf{y}_k (\in Y_N)$ there corresponds a definite element $\mathbf{v}_k$ of the space W_0^N (k = 1, 2,...n),

$$\mathbf{v}_k = \left\{\sum_{i=1}^{N} y_{2i-1,k}\varphi_i, \ \sum_{i=1}^{N} y_{2i,k}\psi_i\right\}. \tag{2.4}$$

The vectors $\mathbf{v}_k$ form a basis in W_0^N, and therefore every element $\mathbf{u} \in W_0^N$ has the form

$$\mathbf{u} = \sum_{i=1}^{n} c_i\mathbf{v}_i. \tag{2.5}$$

We denote by Φ_k the solution of problem (1.1) with boundary conditions given by the vector $\mathbf{v}_k$ (2.4). On the free surface Σ_i(i = 1, 2) the displacement potential Φ_k goes over into the function $\Psi_{i,k}$ in which the vectors $\mathbf{v}_k$ and $\mathbf{\Psi}_k = \{\Psi_{1,k}, \Psi_{2,k}\}$ $(1 \le k \le n)$ are connected by the relation (1.3).

Substituting (2.5) into (1.9), we obtain

$$\omega^2 = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}c_ic_k \left[\rho \sum_{j=1}^{n}\sum_{l=1}^{n} g_{jl}c_jc_l\right]^{-1}, \tag{2.6}$$

where

$$a_{ik} = a_{ki} = \sum_{l=1}^{N} \nu_l y_{2l-1,i}\, y_{2l-1,k} + \sum_{l=1}^{N} \mu_l y_{2l,i} y_{2l,k};$$
$$g_{ik} = g_{ki} = \sum_{l=1}^{N}\left[y_{2l-1,i}\int_{\Sigma_1} \Psi_{1,k}\varphi_l d\Sigma_1 + \sum_{l=1}^{N} y_{2l,i}\int_{\Sigma_2} \Psi_{2,k}\psi_l d\Sigma_2\right]. \tag{2.7}$$

Both quadratic forms are positive-definite and, therefore [10], there is a linear transformation of the variables $\{c_1, c_2, \ldots, c_n\}$:

$$c_i = \sum_{j=1}^{n} U_{ij} x_j, \tag{2.8}$$

which gives Eq. (2.6) the form

$$\omega^2 = \sum_{k=1}^{n} \lambda_k x_k^2 \left[\rho \sum_{i=1}^{n} x_i^2\right]^{-1}. \tag{2.9}$$

Here the $\{\lambda_k\}$ are arranged in order of increasing roots of the equation [10]

$$\det \|a_{ik} - \lambda g_{ik}\| = 0.$$

Solving (1.10), the problem of minimizing Eq. (2.9), we obtain approximate values of the frequencies $w_{j,n}$ and the modes $\mathbf{z}_{j,n}$ of the natural vibrations of the volume Q:

$$\omega_{j,N} = \sqrt{\lambda_j}, \; \mathbf{z}_{j,N} = \sum_{i=1}^{n} U_{ij} \mathbf{v}_i, \; j = 1, 2, \ldots, n.$$

§ 3. Let us consider a volume Q of an ideal liquid in the form of a sphere of radius R and neglect gravity ($\Pi \equiv 0$). We assume that the liquid sphere is in contact with the periphery of a circular opening of radius $r < R$ without changing its shape. The circle of contact divides the surface of the sphere into two parts Σ_1, Σ_2 (Fig. 2). We denote the half-angle of the spherical segment Σ_i ($i = 1, 2$) by β_i. We set $\beta_1 = \beta$ (Fig. 2) and then $\beta_2 = \pi - \beta$.

By assuming that the wettability of the periphery of the opening is complete, the boundary conditions (1.6) have the form

$$u_1|_{L_1} = u_2|_{L_2} = 0.$$

We assume that the surface tensions on Σ_1 and Σ_2 are the same ($\sigma_1 = \sigma_2 = \sigma$). Then it can be shown [9] that for all values of β different from $\pi/2$ the inequality (1.7) is satisfied and the volume Q is in stable equilibrium.

On each segment Σ_i ($i = 1, 2$) we introduce its own curvilinear coordinate system $\{\varphi, s\}$, where φ is the angle of rotation about the axis of symmetry, and s is the arc length measured along the meridian from the pole of the segment ($s = 0$) to the edge of the opening ($s = R\beta_i$). We consider only axisymmetric oscillations of the liquid sphere Q. The problem of determining the eigenvalues and eigenfunctions of the operator A_i (1.5) on the set D_i^χ takes the form [9]

$$-(\sigma/R^2)(d^2u/d\alpha^2 + \operatorname{ctg} \alpha\, du/d\alpha + 2u) = \lambda u, \; 0 < \alpha < \beta_i, \qquad |u(0)| < +\infty, \; u(\beta_i) = 0, \; i = 1,2. \tag{3.1}$$

The eigenfunctions of problems (3.1) are [11] Legendre functions of the first kind $P_{\gamma k}(\cos\alpha)$, $P_{\eta k}(\cos\alpha)$ ($k = 1, 2, \ldots$), where γ_k and η_k are successive roots of the equations

$$P_\gamma(\cos\beta_1) = 0; \; P_\eta(\cos\beta_2) = 0. \tag{3.2}$$

To simplify the calculations we set ρ, σ, and $R = 1$, which corresponds to the transformation to the dimensionless parameter ω^2 in (2.6). It can be shown that the dimensional frequency of oscillations is related to the dimensionless frequency by the expression

$$\omega^2(\rho, \sigma, R) = (\sigma/\rho R^3)\omega^2(1, 1, 1). \tag{3.3}$$

The eigenvalues ν_k and μ_k of the operators A_1 and A_2 are given by the expressions [11]

$$\nu_k = \gamma_k(\gamma_k + 1) - 2; \; \mu_k = \eta_k(\eta_k + 1) - 2. \tag{3.4}$$

The normalized eigenfunctions $\varphi_k(\alpha)$ and $\psi_k(\alpha)$ of the operators A_1 and A_2 have the form

$$\varphi_k(\alpha) = P_{\gamma_k}(\cos\alpha) N_k^{-1}, \quad \alpha \in (0, \beta_1);$$
$$\psi_k(\alpha) = P_{\eta_k}(\cos\alpha) M_k^{-1}, \quad \alpha \in (0, \beta_2),$$

where

$$N_k^2 = \int_0^{\beta_1} (P_{\nu_k}(\cos\alpha))^2 \sin\alpha d\alpha; \quad M_k^2 = \int_0^{\beta_2} (P_{\eta_k}(\cos\alpha))^2 \sin\alpha d\alpha.$$

We obtain the volumes w_k (2.3) by integrating the functions φ_k and ψ_k over the surface of a sphere of unit radius,

$$w_{2k-1} = 2\pi \int_0^{\beta_1} \varphi_k(\alpha) \sin\alpha d\alpha; \quad w_{2k} = 2\pi \int_0^{\beta_2} \psi_k(\alpha) \sin\alpha d\alpha, \qquad k = 1, 2, \ldots . \tag{3.5}$$

Calculations [9] showed that the numbers w_k were different from zero for any half-angles of the spherical segments, and therefore

$$\mathbf{v}_{2j-1} = \left\{ \frac{w_{2j}\varphi_j}{\sqrt{w_{2j-1}^2 + w_{2j}^2}}; \ \frac{-w_{2j-1}\psi_j}{\sqrt{w_{2j-1}^2 + w_{2j}^2}} \right\}, \quad 1 \leqslant j \leqslant N;$$
$$\mathbf{v}_{2j} = \left\{ \frac{w_{2j}\varphi_{2j+1}}{\sqrt{w_{2j}^2 + w_{2j+1}^2}}; \ \frac{-w_{2j+1}\psi_j}{\sqrt{w_{2j}^2 + w_{2j+1}^2}} \right\}, \quad 1 \leqslant j \leqslant N-1 \tag{3.6}$$

can be taken as the basis vectors $\mathbf{v}_k$ (2.4) of the space W_0^N.

The fundamental system of solutions of Eq. (2.2) corresponding to the vectors (3.6) consists of the elements

$$y_{ik} = \begin{cases} 0, \ i < k \quad \text{or} \quad i > k+1 \\ (-1)^{k+1} w_{k+1} (w_k^2 + w_{k+1}^2)^{-1/2}, \quad i = k \\ (-1)^k w_k (w_k^2 + w_{k+1}^2)^{-1/2}, \quad i = k+1, \end{cases} \qquad i = 1, 2 \ldots, 2N; \ k = 1, \ldots, 2N-1. \tag{3.7}$$

By substituting into (2.7) and using (3.3) and (3.7) we obtain for the coefficients $a_{ik} (1 \le i, k \le 2N-1)$

$$a_{ik} = a_{ki} = 2\pi \sum_{l=1}^{N} (\nu_l y_{2l-1,i} y_{2l-1,k} + \mu_l y_{2l,i} y_{2l,k}). \tag{3.8}$$

In order to determine the elements of the matrix g_{ik} (2.7) we construct solutions of problem (1.1) with boundary conditions given by the vectors (3.6). Let θ be the angle measured along the meridian of the sphere ($R=1$) from the pole of the segment Σ_1 ($\theta = 0$) to the pole of the segment Σ_1 ($\theta = \pi$). Using the known [5] solution of the inner Neumann problem (1.1) for a sphere we obtain an expression for the displacement potential Φ_k on the surface:

$$\Phi_k(1, \theta) = \sum_{l=1}^{\infty} (1 + 1/2l) P_l(\cos\theta) \int_0^{\pi} f_k(\alpha) P_l(\cos\alpha) \sin\alpha d\alpha, \qquad k = 1, 2, \ldots, 2N-1, \tag{3.9}$$

where the P_l ($\cos\alpha$) are Legendre polynomials [5, 11], and $f_k(\alpha)$ is the disturbance of the surface of sphere Q given by the vector $\mathbf{v}_k$ (3.6)

$$f_k(\theta) = \begin{cases} v_{1,k}(\theta), \ \theta \in (0, \beta), \\ v_{2,k}(\pi - \theta), \ \theta \in (\beta, \pi). \end{cases} \tag{3.10}$$

We introduce the set of numbers ($j = 1, 2, \ldots, N$; $l = 1, 2, \ldots$)

$$p_{j,l} = \int_0^{\beta} \varphi_j(\alpha) P_l(\cos\alpha) \sin\alpha d\alpha;$$
$$q_{j,l} = \int_0^{\pi-\beta} \psi_j(\alpha) P_l(\cos\alpha) \sin\alpha d\alpha. \tag{3.11}$$

Noting that

$$q_{i,l} = (-1)^l \int_{\pi-\beta}^{\pi} \psi_j(\pi - \alpha) P_l(\cos\alpha) \sin\alpha d\alpha,$$

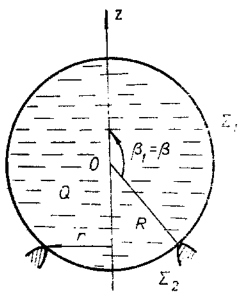

Fig. 2

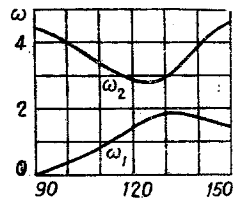

Fig. 3

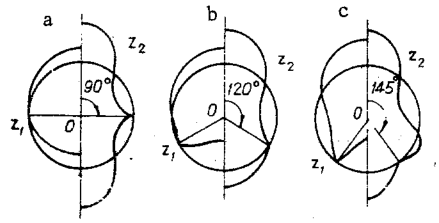

Fig. 4

and using for $\mathbf{v}_k$ in (3.10) the expression (2.4), Eq. (3.9) can be written in the form (k=1, 2,..., 2N−1)

$$\Phi_k(1, \theta) = \sum_{l=1}^{\infty} (1 + 1/2l) V_{k,l} P_l(\cos\theta),$$

where

$$V_{k,l} = \sum_{j=1}^{N} (y_{2j-1,k} p_{j,l} + (-1)^l y_{2j,k} q_{j,l}).$$

Hence, by evaluating the integrals for the coefficients g_{ik} in Eq. (2.7) over the surface of a unit sphere we obtain (i, k=1, 2,..., 2N−1)

$$g_{ik} = g_{ki} = 2\pi \sum_{l=1}^{\infty} (1 + 1/2l) V_{il} V_{kl}. \tag{3.12}$$

Let us fix a certain half-angle β ($>\pi/2$) of the spherical segment Σ_1 (Fig. 2). Specifying the number N of functions φ_k, ψ_k on the surfaces Σ_1, Σ_2, and performing the necessary calculations by using the formulas given above, we find the coefficients a_{ik} (3.8) and g_{ik} (3.12). By minimizing (2.6) with n=2N−1 we obtain the approximate values of the dimensionless frequencies $\omega_{j,N}$ and the corresponding axisymmetric oscillatory modes $\mathbf{z}_{j,N}$ of sphere Q. The transformation to dimensional frequencies is given by (3.3).

The calculations were performed by computer. Tabulated values [9] of the roots of Eqs. (3.2) and volumes (3.5) were used. A good approximation of the first two oscillatory modes (for $90° < \beta < 150°$) was obtained by taking N=4. The quantities $g_{j,l}$ and $p_{j,l}$ (3.11) (j=1, 2, 3, 4; l=1, 2,..., m) were calculated by numerical integration with a check on accuracy. The number m of Legendre polynomials taken into account in (3.12) varied from 10 to 50 as β was increased. The method of rotations was used to minimize Eq. (2.6) to the form (2.8) and to calculate the roots of Eq. (2.9).

Figure 3 shows the dependence of the first two frequencies ω_1 and ω_2 of axisymmetric natural oscillations of liquid sphere Q on the angle of fixation $\beta \in (90, 150°)$.

Figure 4a, b, c shows the natural oscillatory modes $\mathbf{z}_1$, $\mathbf{z}_2$ of sphere Q for angles β = 90, 120, and 145°, respectively. Calculations showed that the first mode of axisymmetric oscillations of the sphere has the same sign on the smaller segment Σ_2 for all values of the angle β. On the larger segment Σ_1 the first mode $\mathbf{z}_1$ has the same sign for $\beta \in (90°, 115°)$; for $\beta > 115°$ there is one change of sign. The second mode $\mathbf{z}_2$ of natural oscillations for $\beta < 108°$ changes sign once on the smaller segment, and for $\beta > 108°$ the second mode has the same sign on Σ_2 (Fig. 4b, c). On segment Σ_1 for $\beta \in (90°, 139°)$ the second oscillatory mode changes sign once (Fig. 4a, b); for $\beta > 139°$ (Fig. 4c) the second mode changes sign twice.

In conclusion, the author thanks F. L. Chernous'ko for posing the problem and for his attention to the work.

LITERATURE CITED

1. N. N. Moiseev and F. L. Chernous'ko, "Problems of the oscillations of a liquid acted upon by surface tension forces," Zh. Vychisl. Mat. Mat. Fiz., No. 6 (1965).
2. N. D. Kopachevskii, "Hydrodynamics in weak force fields. Small oscillations of an ideal liquid," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2 (1966).
3. N. D. Kopachevskii, "Oscillations of immiscible liquids," Zh. Vychisl. Mat. Mat. Fiz., No. 5 (1973).
4. G. N. Mikishev and B. N. Rabinovich, Dynamics of a Rigid Body Containing Cavities Partially Filled with Liquid [in Russian], Mashinostroenie, Moscow (1968).
5. V. S. Vladimirov, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1967).
6. V. Blyashke, Differential Geometry [in Russian], ONTI, Moscow–Leningrad (1935).
7. A. D. Tyuptsov, "Hydrostatics in weak force fields. Stability of equilibrium forms of the surface of a liquid," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2 (1966).
8. S. G. Mikhlin, Variational Methods in Mathematical Physics [in Russian], Nauka, Moscow (1970).
9. V. R. Orel, "Stability of an incompressible fluid acted upon by surface tension forces. Case of a doubly connected equilibrium surface," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1974).
10. F. R. Gantmacher, The Theory of Matrices, Chelsea, New York (1959).
11. E. W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Chelsea, New York (1955).

DYNAMICS OF A DIVERGING LIQUID MENISCUS IN A CAPILLARY, TAKING INTO ACCOUNT THE SPECIFIC PROPERTIES OF THIN FILMS

B. V. Zheleznyi

UDC 532.68

The theory of the diverging meniscus of a Newtonian liquid for capillary flow conditions at low meniscus velocities, in which the thermodynamic and rheological features of thin wetting films appear, is set forth. Two cases are considered: thermodynamically stable wetting film with high viscosity in the boundary layer on a completely wetted solid surface and a thermodynamically unstable film on a conditionally wetted solid surface exhibiting a liquid slip effect.

The relation between the thickness h* of the film left on the walls of the cylindrical capillary behind a diverging liquid meniscus and the rate v at which the meniscus travels is determined when studying the properties of wetting films in the capillary method [1]. Extrapolation of h_* (v) to zero velocity makes it possible to find the thickness of equilibrium films with a meniscus in capillaries of various radii R and to thereby determine the basic thermodynamic characteristic of equilibrium wetting films – the wedging pressure isotherm. Moreover, h_* (v) provides information about the rheological properties of wetting films. A theory of the diverging meniscus that would take into account the specific properties of thin films is necessary in order to interpret this information and to correctly extrapolate h_* (v) to zero velocity.

The dynamics of the diverging meniscus of a wetting liquid has been previously considered under the assumption that the film deposited on a solid film surface exhibits the properties of a bulk liquid phase (the viscosity coefficient η_0 and coefficient of surface tension σ are given by tables) [2-4]. Various methods have yielded the equation

$$\sigma d^3h/dl^3 = 3\eta_0 v(1/h^2 - h_*/h^3), \tag{1}$$

which describes steady flow in one direction in a flat film of a Newtonian liquid on a plane (or circular cylindrical) solid surface if flow occurs only due to capillary forces (capillary flow regime). In Eq. (1) h_* is the

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 72-82, May-June, 1976. Original article submitted April 15, 1975.